INTRODUCTION TO FREE-FORM CURVES AND FREE-FORM SURFACES

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CONTENTS

LIST OF ABBREVIATIONS

3D	Three-Dimensional
B-Spline	Basis Spline
CAD	Computer Aided Design
CAD/CAM	Interconnected CAD/CAM systems
CAE	Computer Aided Engineering
CAM	Computer Aided Machining
FFC	Free-Form Curves
FFS	Free-Form Surfaces
NURBS	Non-Uniform Rational Basis Spline
RE	Reverse Engineering

INTRODUCTION

Free-Form Curves (FFC) and Free-Form Surfaces (FFS) play a fundamental role in the field of Computer Aided Design (CAD) and are widely used in various industries such as automotive, aerospace, architecture, and animation. These mathematical constructs enable the creation of complex, smooth shapes that are not confined to simple geometric forms, providing designers and engineers with unparalleled flexibility and precision (Morovič, Kuruc 2024).

This report explores the essential concepts of FFC and FFS, with a focus on their basic mathematical foundations. The discussion begins with an short overview of FFC, covering Bézier curves, B-splines, and NURBS (Non-Uniform Rational B-Splines) – three fundamental types that have revolutionized modern design and modeling techniques. It then extends to FFS, delving into T-splines, B-spline surfaces, and NURBS surfaces, which allow for the modeling of intricate Three-Dimensional (3D) forms.

The aim of this report is to provide an accessible introduction to these topics, catering to a broader audience, including those without a technical background. This report seeks to demystify free-form design and showcase its transformative impact on modern industries.

At the forefront of the development of FFC are two pivotal figures: **Pierre Bézier** and **Paul de Faget de Casteljau**. **Pierre Bézier** (1910 – 1999), a French engineer, is celebrated for his work at **Renault**, where he developed **Bézier curves**, which have become a cornerstone of modern CAD and 3D modelling (Bézier 1986). **Paul de Faget de Casteljau** (1930 – 2022), a mathematician working at **Citroën**, independently created the **De Casteljau algorithm**, a robust method for evaluating Bézier curves (Casteljau 1986). While their approaches were developed independently, their combined contributions laid the foundation for the precise mathematical tools that drive free-form design today.

Fig. I.1 *Pierre Étienne Bézier (1910 – 1999)*
(Journals Open Edition 2025)

Fig. I.2 *Paul de Faget de Casteljau (1930 – 2022)*
(Solid Modeling Association 2025)

1 FREE-FORM CURVES

Non-Uniform Rational B-Splines (NURBS) were developed from the work of **Pierre Étienne Bézier** (1910 – 1999) and are also known as **Bézier curves and surfaces**. Bézier's initial work, which began in the automotive industry, involved the use of splines to design the **bodywork of Renault cars** in the late **1960s and early 1970s**. Due to their smoothness and flexibility, **B-spline curves** were developed as a natural extension. **NURBS** provide the ability to represent a broad variety of curves and surfaces, ranging from straight lines to exact circles, as well as highly detailed sculpted surfaces. Additionally, **NURBS** allow these elements to be embedded within a smoothly sculpted surface, which is crucial for creating Computer Aided Design (CAD) in industries such as automotive, aerospace, and consumer products with complex surface structures (Rogers 2001).

While **Bézier curves** are widely used for FFC, more complex designs often require **B-spline curves**, which offer greater control over the shape. **B-spline curves** can be created through curve subdivision, and **NURBS** provide further refinement through **weights** assigned to the control points. NURBS make it possible to design intricate shapes in both planar and 3D space (Pottman *et al.* 2007).

As shown in ***Fig. 1.1***, different types of FFC can result different shapes with the same control polygons and points (Urban 2024).

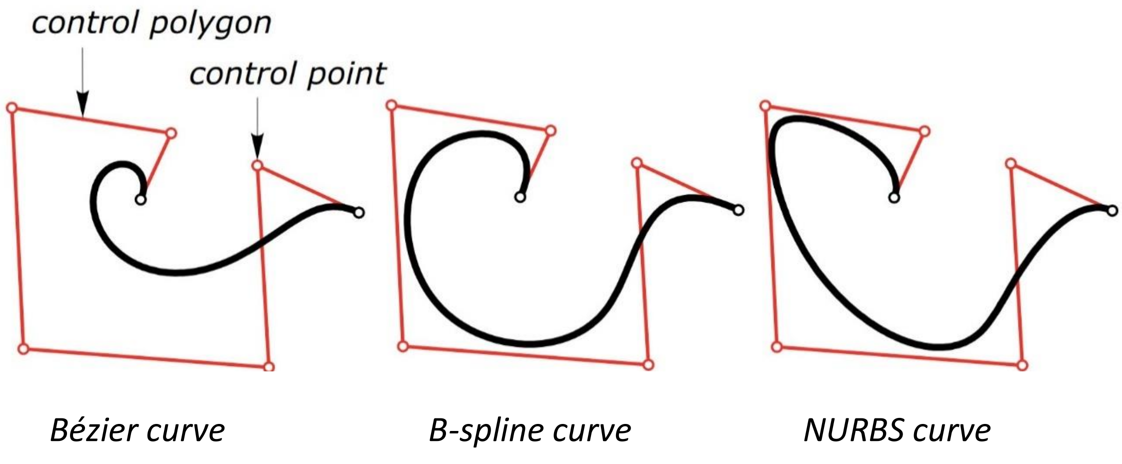


Fig. 1.1 *Free-form curves in design.*
Compared with the same control polygons (Pottman et al. 2007).

1.1 Bézier curves

In chronological sequence, Bézier curves were created first, and from them, NURBS were developed as a more generalized version of Bézier curves (Rogers 2001), (Urban 2024).

The concept of similar curves was first introduced by **Paul de Faget de Casteljau** (1930 – 2022) during his work at **Citroën**, and later further developed by **Pierre Étienne Bézier** at **Renault**. These curves are parametric and are controlled by a set of control points, $B_0, B_1, \ldots, B_n$. The position of a point P(t) on the curve is determined by **weighted** averages of the coordinates of the control points. The coordinates of the curve points are also functions of a parameter t, which ranges from 0 to 1. The curve begins at t = 0 and ends at t = 1 (Rogers 2001).

The simplest curve is a straight line, it connects two points

$$\boldsymbol{B_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \boldsymbol{B_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \tag{1}$$

The coordinates on line B_0B_1 are given as functions of the parameter *t*

$$\boldsymbol{P}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = (1 - t)\boldsymbol{B_0} + t\boldsymbol{B_1}, \quad t = [0,1] \tag{2}$$

A second degree **(quadratic)** curve is given by three points

$$\boldsymbol{P}(t) = (1 - t)^2\boldsymbol{B_0} + 2(1 - t)t\boldsymbol{B_1} + t^2\boldsymbol{B_2} \tag{3}$$

The next degree curve is defined by four control points, it is called **cubic** Bézier curve

$$\boldsymbol{P}(t) = (1 - t)^3\boldsymbol{B_0} + 3(1 - t)^2t\boldsymbol{B_1} + 3(1 - t)t^2\boldsymbol{B_2} + t^3\boldsymbol{B_3} \tag{4}$$

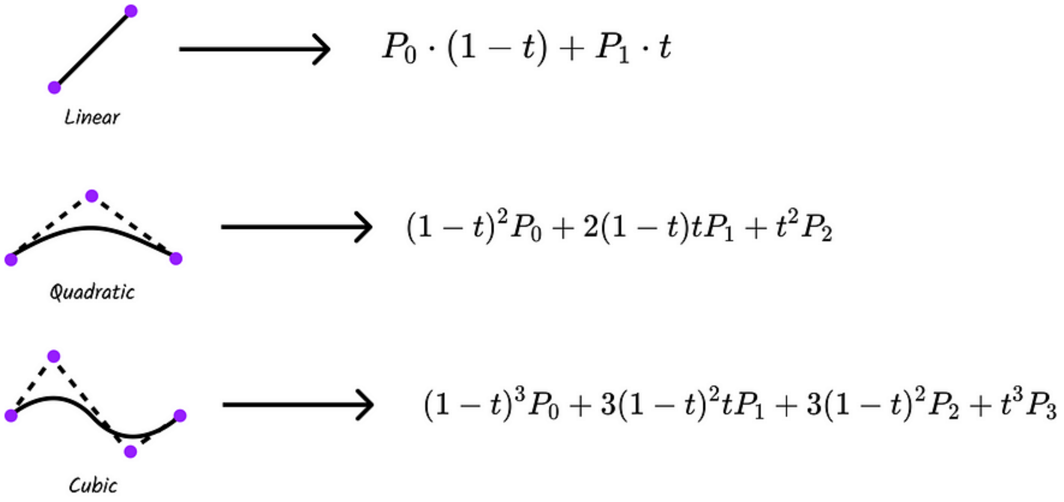


Fig. 1.2 *Basic Bézier curves of different degrees (Melo 2021).*

1.2 B-splines

A more **advanced (sophisticated)** type of curve is the **B-spline**. Unlike Bézier curves, where moving a control point alters the entire curve, B-splines do not have this limitation. They allow for local adjustments by modifying individual control points (Rogers 2001).

As shown in ***Fig. 1.3***, B-spline curves are composed of Bézier curve segments, where each segment is connected with the highest possible smoothness – this can include having matching tangents or curvatures, among other conditions (Pottman *et al.* 2007).

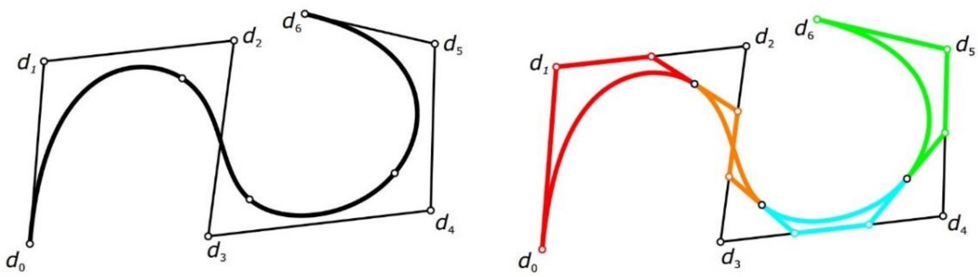


Fig. 1.3 *B-spline curve of degree n=3*
which consists of four Bézier curve segments (Pottman et al. 2007).

Given n+1 control points, $B_1, \ldots, B_{n+1}$, the position vector is

$$P(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t), \quad t_{min} \le t \le t_{max}, \quad 2 \le k \le n+1 \qquad (5)$$

B-splines are preferred over Bézier curves in the design industry due to a feature known as **curvature continuity**, which results in smoother shapes. This property ensures that Bézier segments of a B-spline curve are connected as smoothly as possible. Without this feature, manually connecting Bézier curves would lead to discontinuities or non-tangent connections, which are undesirable in certain designs and detectable to the human eye ***(Fig. 1.4)*** (Pottman *et al.* 2007).

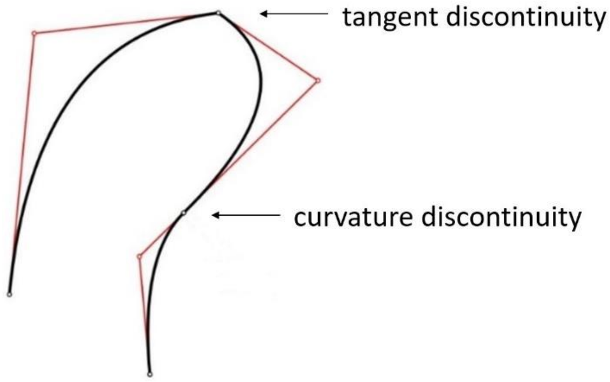


Fig. 1.4 *Tangent and curvature discontinuity (Pottman et al. 2007).*

Although **B-spline curves offer more flexibility than Bézier curves**, they still cannot represent all possible curves, including simple shapes such as circles, hyperbolas, or ellipses.

1.3 NURBS curves

NURBS curves are essentially B-splines with a non-uniform knot vector, although they can also have a uniform knot vector in some cases. When working with NURBS, an additional parameter known as **weights** is introduced. These weights are linked to the control points and contribute to the "rational" nature of NURBS. A NURBS curve is essentially a projection of a B-spline curve into a higher-dimensional space. A NURBS curve is basically placed in dimension d and is central projected of a B-spline placed in a dimension $d+1$ (Pottman *et al.* 2007), (Urban 2024).

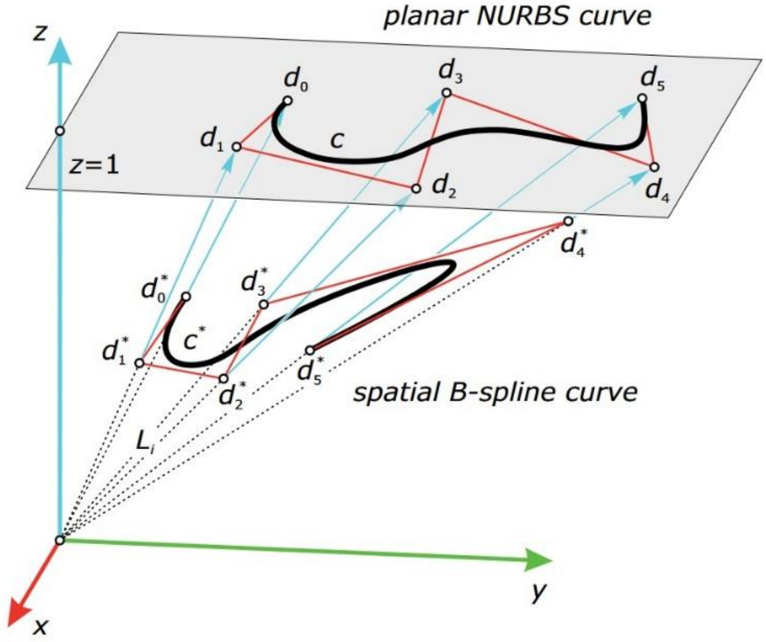


Fig. 1.5 *Projection of a 3-dimensional B-spline curve into a planar NURBS (Pottman et al. 2007).*

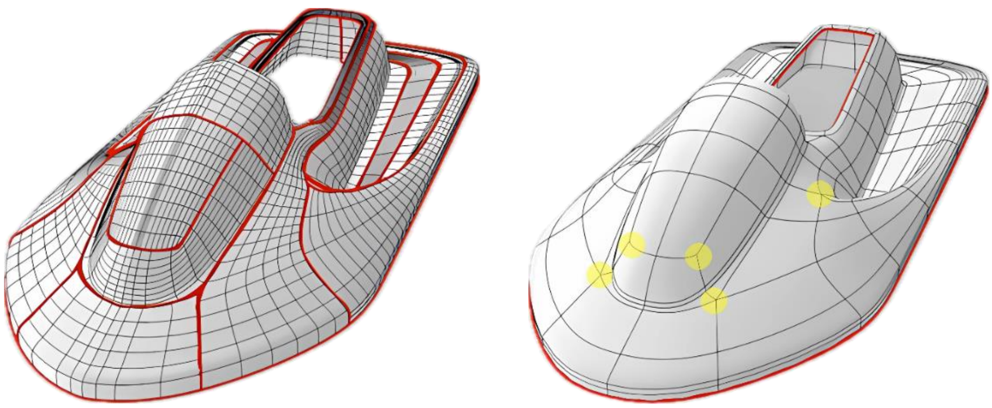
For a planar NURBS curve, the weights correspond to the z-coordinates of the control points of the associated 3D B-spline curve, a principle that also applies to higher dimensions. Adjusting the weight of a control point only affects the local area of the curve, as moving a control point on a B-spline results in localized changes. In this sense, B-splines can be seen as a special case of NURBS where all the weights are equal (Pottman *et al.* 2007).

2 FREE-FORM SURFACES

FFS are complex geometric shapes used extensively in fields such as computer graphics, industrial design, and CAD. Unlike standard geometric forms, they are defined by mathematical representations like Bézier, B-splines, and NURBS, allowing for smooth and flexible shapes. These surfaces enable the creation of highly detailed and aesthetically pleasing models, making them essential for applications ranging from automotive design to animation and product modelling. Their versatility and precision make them a cornerstone in modern design and visualization techniques.

2.1 T-splines

A recently developed design technology called **T-splines** extends NURBS surfaces by allowing local control and minimizing the drawbacks of NURBS. The primary benefit of T-splines is that they facilitate local editing, meaning adding a control point to the mesh does not necessitate the addition of multiple other points. This feature makes T-splines more intuitive for users when designing and editing surface shapes, focusing only on areas requiring detail. When using T-splines, fewer control points are needed to define a surface compared to NURBS, as demonstrated in ***Fig. 2.1.*** Additionally, T-spline surfaces can be converted into NURBS surfaces, ensuring compatibility with existing CAD/CAM systems (Wang 2009).



a) *b)*

Fig. 2.1 *Watercraft modeled as (Sederberg, Sederberg 2010):*
a) *NURBS by 13 surfaces,* ***b)*** *T-spline by 1 surface.*

Similar to NURBS, T-spline surfaces are defined by a control grid, known as a **T-mesh**. A T-mesh allows partial rows or columns of control points to be added, enabling the insertion of a single control point without needing to create an entire row or column. T-meshes can include various types of **T-junctions**, which are categorized based on their connectivity, as shown in ***Fig. 2.2*** (Wang 2009), (Urban 2024).

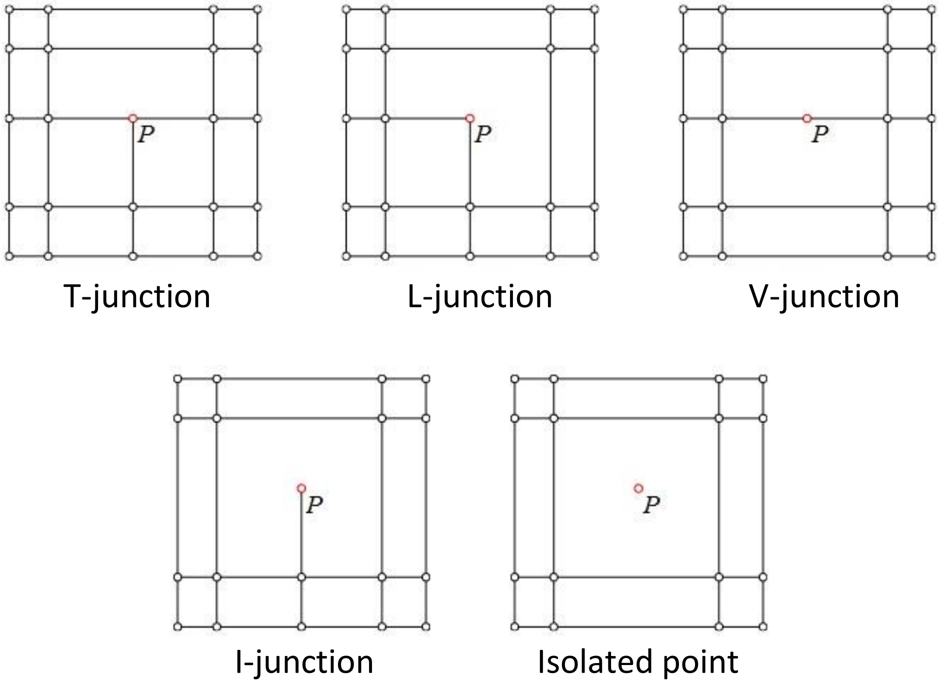


Fig. 2.2 *Different types of T-junctions (Wang 2009).*

2.2 B-Spline and NURBS surfaces

As mentioned earlier, Bézier curves are not well-suited for representing more complex control meshes, and the same is true for Bézier surfaces, which are closely related to Bézier curves. Thus, B-splines offer one possible solution for surface definition. A B-spline can be defined by a quadrilateral control mesh, with the option to adjust the degrees of the *u-curves* and *v-curves*. Another solution is the use of NURBS surfaces, which attach weights to each control point, allowing for changes in the surface shape. The effect of modifying weights is similar to that in NURBS curves, as shown in ***Fig. 2.3*** (Pottman *et al.* 2007).

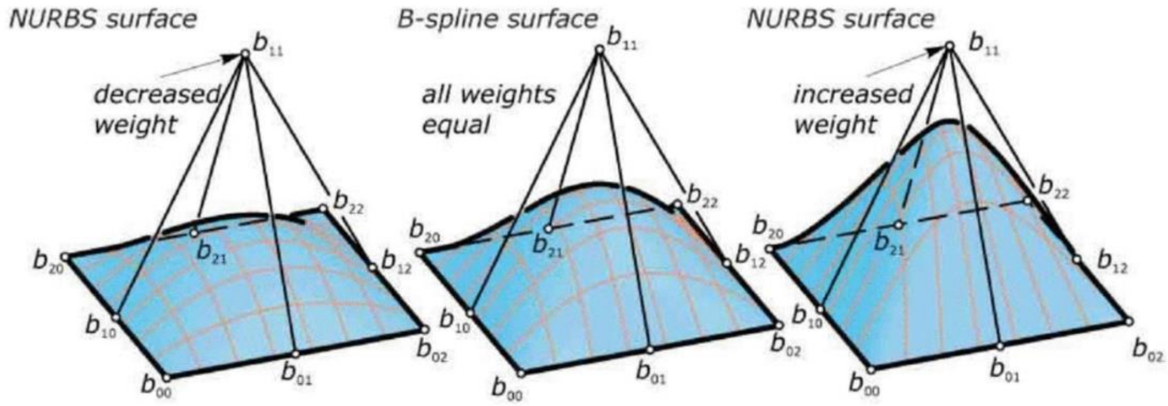


Fig. 2.3 *Effects of changing the weight of a control point (Pottman et al. 2007).*

B-spline and NURBS curves can be constructed in either an open or closed mode. For the closed mode, the polygon must also be closed. A B-spline surface consists of two types of curves, *u-curves* and *v-curves,* which can be either open or closed. This results in three distinct ways the surface can be constructed ***(Fig. 2.4)*** (Pottman *et al.* 2007):

- Open mode for both *u-curves* and *v-curves:* the surface is a four-sided patch;
- Closed mode in one direction (*u* or *v*), open in the second direction: the surface resembles a deformed pipe;
- Closed mode in both directions: the surface appears as a deformed torus.

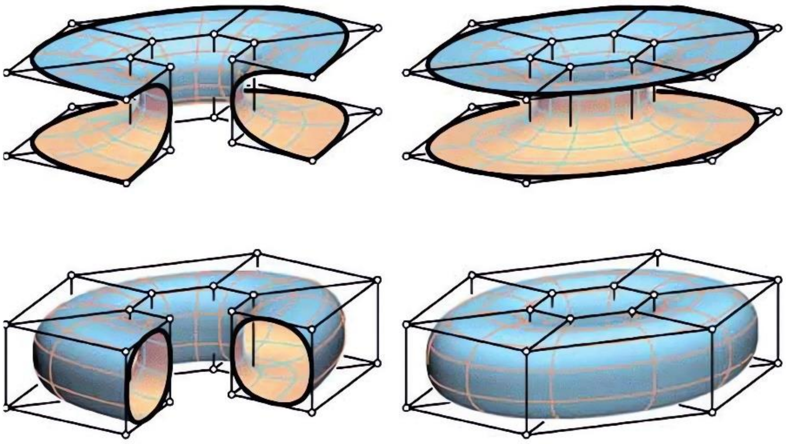


Fig. 2.4 *Different types of surfaces, depending on the chosen mode (Pottman et al. 2007).*

CONCLUSION

In summary, FFC and FFS are fundamental tools in modern design and modelling, offering unmatched flexibility and precision. Bézier curves, B-splines, and NURBS provide a robust foundation for constructing smooth and customizable shapes, making them invaluable in areas like computer graphics and CAD. Similarly, FFS such as T-splines, B-splines, and NURBS surfaces extend this flexibility to three-dimensional modeling, enabling the creation of complex and highly detailed forms. Their mathematical basis ensures accuracy, while their adaptability supports innovation across various industries. Understanding these concepts is essential for anyone working in fields that rely on advanced geometric design.

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